1) 8th Macedonian Mathematical Olympiad of 2000:

Prove that if $m \cdot s = 2000^{2001}$ where m, s are integers; then the equation $mx^2 - sy^2 = 3$ has no integer solution.

2) 2000 Math Olympiad of Ireland:
If $f(x) = 5x^{13} + 13x^5 + 9ax$, find the least positive integer a such that 65 divides $f(x)$ for every integer x.

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1. Introduction

The subject matter of this paper is two Mathematical Olympiad problems from the year 2000.

The first problem, listed below as PROBLEM1, was featured in the 2000 Macedonian Mathematical Olympiad.

The second problem, PROBLEM2, was part of the 13th Irish Mathematical Olympiad of 2000.

Let us state the two problems:

PROBLEM 1

8th MACEDONIAN MATHEMATICAL OLYMPIAD OF 2000

Problem#1: Prove that if $m \cdot s = 2000^{2001}$; where $m, s \in \mathbb{Z}$; then the equation $mx^2 - sy^2 = 3$ has no solution in $\mathbb{Z}$.

PROBLEM2

13 th IRISH MATHEMATICAL OLYMPIAD

May 6, 2000

Time: 6 hours

Problem#3: Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x

In our solution to PROBLEM2 in Section 6, we show that $a = 63$ is the smallest such positive integer.

1. Introduction

PROBLEM 1 is published in the November 2004 issue of the journal CRUX MATHEMATICORUM; issue No 7, Volume 30, on page 414.

PROBLEM 2 is published in the December 2004 issue of CRUX; issue No 8, Volume 30, page 377.

There are five results in this paper; Results 1–5

Result 1 and Result 2 pertain to PROBLEM 1:

Result 1

Let a be an integer not divisible by 3; $a \in \mathbb{Z}$ and $a \not\equiv 0 \pmod{3}$. And let n be a non-negative even integer and k a positive odd integer; $n \in \mathbb{Z}$, $n \geq 0$, $n \equiv 0 \pmod{2}$; $k \in \mathbb{Z}^+$, $k \equiv 1 \pmod{2}$.

Then:

(i) $a^2 \equiv 1 \pmod{3}$

(ii) $a^n \equiv 1 \pmod{3}$

(iii) If $a \equiv 2 \pmod{3}$; $a^k \equiv 2 \pmod{3}$

(iv) If $a \equiv 1 \pmod{3}$; $a^k \equiv 1 \pmod{3}$

1. Introduction

We prove Result1 in Section2. Then, in Section3, using Result1, we present a proof to Result2:

Result 2

Let b and c be fixed (or given) positive integers, with b being congruent to $2 \pmod 3$; and c being odd; $b, c \in \mathbb{Z}^+$, $b \equiv 2 \pmod 3$, and $c \equiv 1 \pmod 2$.
Moreover, let d_1 and d_2 be integer divisors of the integer b^c, such that $d_1 \cdot d_2 = b^c$.
Consider the 2-variable equation,

$$d_1 \cdot x^2 - d_2 \cdot y^2 = 3.$$

Then, this equation has no integer solution

Note that Result2 generalizes PROBLEM 1:
The hypothesis in PROBLEM1 is a special case of the hypothesis of Result 2. Indeed, it is the case $b = 2000 \equiv 2 \pmod 3$ and $c = 2001 \equiv 1 \pmod 2$.

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The next three results are in connection to PROBLEM 2.

Result 3 stated below, is the well known theorem in number theory; known as Fermat's Little Theorem. We state it without proof.

Result 3 (Fermat's Little Theorem)

Let p be an odd prime number; and let a be an integer not divisible by the prime p; $a \in \mathbb{Z}$ and $a \not\equiv 0 \pmod{p}$ (or equivalently, since p is a prime, $g.c.d(a, p) = 1$). Then,

$$a^{p-1} \equiv 1 \pmod{p}.$$

In particular,

(i) For $p = 5$: $\quad a^4 \equiv 1 \pmod{5}$

(ii) For $p = 13$: $\quad a^{12} \equiv 1 \pmod{13}$

We use Result 3 in the solution to PROBLEM 2 in Section 6.

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Next we have,

Result 4

Let S be the set of all integer solution pairs to the 2-variable linear equation,

$$5y = 13x + 8.\ ;$$

$$S = \{(x,y) \mid x,y \in \mathbb{Z};\ \text{and}\ 5y = 13x + 8\}.$$

Then, the set S can be described in terms of one integer parameter t:

$$S = \left\{(x,y) \mid (x,y) = (5t-16,\ 13t-40);\ t \in \mathbb{Z};\ t = 0, \pm 1, \pm 2, \ldots\right\}.$$

The solution set S consists of all pairs (x,y) of the form,

$(x,y) = (5t-16,\ 13t-40)$; where t can be any integer; $t \in \mathbb{Z}$; $t = 0, \pm 1, \pm 2, \pm 3, \ldots$

We prove Result 4 in Section 4.

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In Section 5, using Result 4, we establish Result 5:

Result 5

Let T be the set of all integers v such that $v = 5y = 13x + 8$; for some integers x and y;

$$T = \{v \mid v \in \mathbb{Z} \text{ and } v = 5y = 13x + 8;\ \text{for } x, y \in \mathbb{Z}\}$$

Then, T consists of all integers of the form $65t - 200$; where t can be any integer:

$$T = \left\{v \mid v \in \mathbb{Z} \text{ and } v = 65t - 200;\ t \in \mathbb{Z};\ t = 0, \pm 1, \pm 2, \pm 3, \ldots\right\}$$

In Section 6, using Result 3 and Result 5; we present a solution to PROBLEM 2. We show that the answer to PROBLEM 2 is $a = 63$.

Section 2: Proof of Result 1

(i) Since $a \not\equiv 0 \pmod 3$; we have $a \equiv 1$ or $2 \pmod 3$.

If $a \equiv 1 \pmod 3$, then $a^2 \equiv 1 \cdot 1 \equiv 1 \pmod 3$.

And if $a \equiv 2 \pmod 3$, $a^2 \equiv 2^2 \equiv 2 \cdot 2 \equiv 4 \equiv 1 \pmod 3$ $\blacksquare$

(ii) n is a non-negative integer; $n = 2\rho$, $\rho \in \mathbb{Z}$, $\rho \geq 0$. Since $\rho \geq 0$, it is clear that a^ρ is an integer since $a \in \mathbb{Z}$; $a^\rho \in \mathbb{Z}$; and $a^\rho \not\equiv 0 \pmod 3$; since $a \not\equiv 0 \pmod 3$

We have, $a^n = a^{2\rho} = (a^\rho)^2 \equiv 1 \pmod 3$ ✓ $\blacksquare$

by part (i), in view of $a^\rho \not\equiv 0 \pmod 3$

(iii) Since k is an odd positive integer; we have $k = 2w+1$; $w \in \mathbb{Z}$ and $w \geq 0$.

We have: $a^k = a^{2w+1} = a^{2w} \cdot a \equiv 1 \cdot a \equiv a \pmod 3$, $\equiv 2 \pmod 3$;

Since $2w$ is an even non-negative integer and thus, by part (i), $a^{2w} \equiv 1 \pmod 3$; in view of $a \equiv 2 \not\equiv 0 \pmod 3$. $\blacksquare$

(iv) We have, $a^k = a^{2w} \cdot a \equiv 1 \cdot a \equiv a \equiv 1 \pmod 3$ $\blacksquare$

by part (i)

Section 3: Proof of Result 2

We have:

$$\left\{\begin{array}{l} b, c \in \mathbb{Z}^+;\ b \equiv 2 \pmod 3 \text{ and } c \equiv 1 \pmod 2. \\ \text{And },\ d_1 \cdot d_2 = b^c;\ \text{with } d_1, d_2 \in \mathbb{Z}. \\ \text{And the equation,} \\ \qquad d_1 \cdot x^2 - d_2 \cdot y^2 = 3;\ \text{and } x, y \in \mathbb{Z} \end{array}\right\} (1)$$

First, observe that the equation in (1) cannot have a solution with $x \cdot y \equiv 0 \pmod 3$; a solution with at least one of x, y being divisible by 3.

To see this, note that from (1) it follows that

$$\left\{\begin{array}{l} b^c \equiv 2^c \pmod 3 \\ c, b \in \mathbb{Z}^+,\ b \equiv 2 \pmod 3,\ c \equiv 1 \pmod 2 \end{array}\right\} (2)$$

But then, since c is an odd positive integer; it follows from Result 1(iii) that,

$$\left\{ 2^c \equiv 2 \pmod 3 \right\} \quad (3)$$

From (3) and (2),

$$\left\{\begin{array}{l} b^c \equiv 2 \pmod 3; \\ \text{And since (by (1)), } d_1 \cdot d_2 = b^c; \\ \text{we have, } \qquad d_1 \cdot d_2 \equiv 2 \pmod 3 \\ \text{Thus, in particular, } \quad d_1 d_2 \not\equiv 0 \pmod 3 \\ \text{And since 3 is a prime; } d_1 \not\equiv 0 \text{ and } d_2 \not\equiv 0 \pmod 3 \end{array}\right\} (4)$$

Section 3: Proof of Result 2

Now, back to the observation that the equation in (1) cannot have an integer solution (x,y) with $xy \equiv 0 \pmod{3}$.

Indeed, if $x \equiv 0 \pmod{3}$, then $-d_2 \cdot y^2 \equiv 0 \pmod{3}$; as it easily follows from (1).

But then, since $d_2 \not\equiv 0 \pmod{3}$ by (4); $-d_2 \cdot y^2 \equiv 0 \pmod{3}$ implies that 3 divides y^2; and thus 3 divides y.

Same argument if $y \equiv 0 \pmod{3}$: (1) implies that x must also be divisible by 3 (since $d_1 \not\equiv 0 \pmod{3}$ by (4))

We have shown that if $xy \equiv 0 \pmod{3}$, then $x \equiv y \equiv 0 \pmod{3}$; which implies that the left-hand side of the equation in (1) is divisible by 9; an impossibility, since the right-hand side of (1) is equal to 3.

We have shown that (1) implies:

$$\left\{ xy \not\equiv 0 \pmod{3} \underset{\text{since 3 is prime}}{\Longleftrightarrow} x \not\equiv 0 \pmod{3} \text{ and } y \not\equiv 0 \pmod{3} \right\} \quad (5)$$

Therefore, (5) further implies that

$$\left\{ x^2 \equiv y^2 \equiv 1 \pmod{3} \right\} \quad (6)$$

Section 3: Proof of Result 2

From (6) and (1) we further obtain,

$$d_1 \cdot 1 - d_2 \cdot 1 \equiv 0 \pmod 3;$$

$$\left\{ d_1 \equiv d_2 \pmod 3 \right\} \quad (7)$$

Hence, it follows from (7) that,

$$\left\{ \begin{array}{l} d_1 \cdot d_2 \equiv d_1 \cdot d_1 \equiv (d_1)^2 \equiv 1 \pmod 3, \\ \text{since } d_1 \not\equiv 0 \pmod 3 \text{ (by (4)) and} \\ \text{by } \underline{\text{Result 1(i)}} \end{array} \right\} \quad (8)$$

Clearly, we have a contradiction:

In (8) we have $d_1 \cdot d_2 \equiv 1 \pmod 3$,

while in (7) we have $d_1 \cdot d_2 \equiv 2 \pmod 3$.

This proves that the equation (1) has no integer solution. ∎

Section 4: Proof of Result 4

We have the equation,

$$\left\{\begin{array}{l} 5y = 13x + 8; \\ x, y \in \mathbb{Z} \end{array}\right\} \quad (1)$$

(a) First, sufficiency: We show that every pair (x,y) of the form $(x,y) = (5t-16, 13t-40)$; $t \in \mathbb{Z}$; is a solution to (1)

Indeed: $5y = 5(13t-40) = 65t - 200$

And $13x+8 = 13(5t-16)+8 = 65t - (13)(16)+8$

$$= 65t - 208 + 8$$

$$= 65t - 200 = 5y \checkmark \quad \blacksquare$$

(b) Next, necessity: We show that if (x,y) is a solution pair to (1); then $x = 5t-16$ and $y = 13t-40$; for some $t \in \mathbb{Z}$.

We have $(1) \Leftrightarrow \left\{\begin{array}{l} y = \dfrac{13x+8}{5} \\ x, y \in \mathbb{Z} \end{array}\right\}$

$$\Leftrightarrow \left\{\begin{array}{l} y = \dfrac{10x}{5} + \dfrac{3x+8}{5} \\ x, y \in \mathbb{Z} \end{array}\right\};$$

or equivalently,

Section 4: Proof of Result 4

$$\begin{cases} y = 2x + \dfrac{3x+8}{5}; \\ y \in \mathbb{Z} \text{ and } x \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} y = 2x + \dfrac{3x+8}{5}; \\ y \in \mathbb{Z},\ x \in \mathbb{Z}, \text{ and } \dfrac{3x+8}{5} = z \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} y = 2x + z; \\ y \in \mathbb{Z},\ x \in \mathbb{Z},\ z \in \mathbb{Z}; \text{ and } 3x = 5z - 8 \end{cases}$$

$$\iff \begin{cases} y = 2x+z; \quad y \in \mathbb{Z}, x \in \mathbb{Z}, z \in \mathbb{Z}; \\ \text{And} \quad x = \dfrac{5z-8}{3} = 2z - \left(\dfrac{z+8}{3}\right); \\ \text{and also} \quad \dfrac{z+8}{3} = t \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} y = 2x+z; \quad y, x, z \in \mathbb{Z}; \text{ and} \\ z = 3t-8; \quad t \in \mathbb{Z}; \text{ and } x = 2z - t \end{cases}$$

$$\iff \begin{cases} y = 2x+z, \quad z = 3t-8, \quad x = 2z-t; \\ \text{with } t \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} y = 2(2z-t)+z, \quad z = 3t-8, \quad x = 2(3t-8)-t; \\ t \in \mathbb{Z} \end{cases}$$

$$\begin{cases} y = 4z - 2t + (3t-8) = 4(3t-8) - 2t + 3t - 8 = 13t - 40; \\ z = 3t-8, \quad x = 5t-16; \quad t \in \mathbb{Z} \end{cases}$$

$$(x,y) = (5t-16,\ 13t-40) \checkmark$$

Section 5: Proof of Result 5

We have the set,

$$T = \{ v \mid v \in \mathbb{Z} \text{ and } v = 5y = 13x + 8; \text{ for } x, y \in \mathbb{Z} \}$$

We wish to show that every element of v is of the form, $v = 65t - 200$, for some $t \in \mathbb{Z}$

First, let us show that if $v = 65t - 200$, for $t \in \mathbb{Z}$. Then v is an element of T; $v \in T$.

Indeed: $v = 65t - 200 = 5(13t - 40) = 65t - 208 + 8$
$= 13(5t - 16) + 8;$

Therefore, $v = 5y = 13x + 8$; where $y = 13t - 40 \in \mathbb{Z}$ and $x = 5t - 16 \in \mathbb{Z}$.

Conversely, suppose that $v \in T$. Then, by the definition of the set T; we have
$v = 5y = 13x + 8$; for some $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

By Result 4, the integer pair (x, y) is of the form; $(x, y) = (5t - 16, 13t - 40)$, for some integer t; $t \in \mathbb{Z}$.

Therefore, $v = 5y = 5(13t - 40) = 65t - 200$
$= 13x + 8 = 13(5t - 16) + 8 = 65t - 200$ ✓

The proof is complete ∎

Section 6: Solution to PROBLEM2

Let A be the set of all integers a with the property that $f(x) = 5x^{13} + 13x^5 + 9ax$ is divisible by 65, for all $x \in \mathbb{Z}$;

$$A = \left\{ a \mid a \in \mathbb{Z} \text{ and } f(x) = 5x^{13} + 13x^5 + 9ax \text{ is divisible by } 65 \text{ for all } x \in \mathbb{Z}. \right\} \quad (1)$$

Since $65 = 5 \cdot 13$, and 5 and 13 are both primes; it follows that:

$$\left\{ \begin{array}{l} f(x) \equiv 0 \pmod{65} \\ \text{For all } x \in \mathbb{Z} \end{array} \right. \Longleftrightarrow \left(\begin{array}{l} \text{For all } x \in \mathbb{Z} \\ f(x) \equiv 0 \pmod{5} \text{ and} \\ f(x) \equiv 0 \pmod{13} \end{array} \right\} \quad (2)$$

In view of (2) and (1); we have:

$$A = \left\{ a \mid a \in \mathbb{Z} \text{ and } f(x) = 5x^{13} + 13x^5 + 9ax \equiv 0 \pmod{5} \text{ for all } x \in \mathbb{Z}. \text{ And also, } f(x) \equiv 0 \pmod{13}; \text{ for all } x \in \mathbb{Z} \right\} \quad (3)$$

We will prove that the set A precisely (contains) consists all the integers of the form, $\underline{a = 65t - 197}$.

Section 6 : Solution to PROBLEM 2

To do so, let us consider each of the two congruences ; the congruence $f(x) \equiv 0 \pmod{5}$, and the congruence $f(x) \equiv 0 \pmod{13}$.

First :
$$\left\{\begin{array}{l} f(x) \equiv 0 \pmod{5} \\ \text{for all } x \in \mathbb{Z} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \overset{\equiv 0 \pmod 5}{5x^{13}} + 13x^5 + 9ax \equiv 0 \pmod{5} \\ \text{for all } x \in \mathbb{Z} \end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l} 0 + 3x^5 + 9ax \equiv 0 \pmod{5} \\ \text{for all } x \in \mathbb{Z} \end{array}\right\}$$

since $5 \equiv 0 \pmod{5}$, $9 \equiv 4 \pmod{5}$ and $13 \equiv 3 \pmod{5}$

$$\Longleftrightarrow \left\{\begin{array}{l} x \cdot (3x^4 + 4a) \equiv 0 \pmod{5} \\ \text{for all } x \in \mathbb{Z} \end{array}\right\}$$

Clearly the last congruence is true if $x \equiv 0 \pmod{5}$ and a is any integer. For the last congruence to be true for all $x \in \mathbb{Z}$; it is necessary and ~~sufficien~~ sufficient that

$$\left\{\begin{array}{l} x \cdot (3x^4 + 4a) \equiv 0 \pmod{5} \\ \text{For all } x \in \mathbb{Z} \quad ; \text{with } x \not\equiv 0 \pmod{5} \end{array}\right\}$$

Or equivalently (since $x \not\equiv 0 \pmod{5}$),

Section 6: Solution to PROBLEM 2

$$\left\{\begin{array}{l} 3x^4 + 4a \equiv 0 \pmod 5; \\ \text{For all } x \in \mathbb{Z} \text{ with } x \not\equiv 0 \pmod 5 \end{array}\right\}$$

By Result 3 (Fermat's Little Th.) (ii); we have $x^4 \equiv 1 \pmod 5$. Thus the above congruence is equivalent to,

$$\left\{ 3 + 4a \equiv 0 \pmod 5 \right\}$$

$$\Longleftrightarrow \left\{ 4a \equiv -3 \equiv 2 \pmod 5 \right\}$$

$$\Longleftrightarrow \left\{ (-1)a \equiv 2 \pmod 5 \right.$$

$$\Longleftrightarrow \left\{ a \equiv -2 \equiv 3 \pmod 5 \right\}$$

We have shown that:

$$\left\{\begin{array}{l} f(x) = 5x^{13} + 13x^5 + 9ax \equiv 0 \pmod 5 \\ \text{For all } x \in \mathbb{Z}; \quad a \in \mathbb{Z} \end{array}\right\} \Longleftrightarrow \left\{ a \equiv 3 \pmod 5 \right\} \qquad (4)$$

Section 6: Solution to PROBLEM2

We work similarly in the case of the second congruence:

$$\left\{\begin{array}{l} f(x) = 5x^{13} + 13x^5 + 9ax \equiv 0 \pmod{13}; \\ \text{For all } x \in \mathbb{Z};\ a \in \mathbb{Z} \end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l} f(x) = x \cdot (5x^{12} + 13x^4 + 9a) \equiv 0 \pmod{13} \\ \text{for all } x \in \mathbb{Z};\ a \in \mathbb{Z} \end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l} f(x) = x(5x^{12} + 0 \cdot x^4 + 9a) \equiv 0 \pmod{13} \\ \text{for all } x \in \mathbb{Z};\ a \in \mathbb{Z} \end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l} f(x) \equiv 5x^{12} + 9a \equiv 0 \pmod{13} \\ \text{for all } x \in \mathbb{Z},\ x \not\equiv 0 \pmod{13},\ a \in \mathbb{Z} \end{array}\right\}$$

$$\underset{\text{by Result2 (iii)}}{\Longleftrightarrow} \left\{\begin{array}{l} f(x) \equiv 5 + 9a \equiv 0 \pmod{13} \\ \text{for all } x \in \mathbb{Z},\ x \not\equiv 0 \pmod{13} \\ a \in \mathbb{Z} \end{array}\right\}$$

$$\Longleftrightarrow \left\{\begin{array}{l} 9a \equiv -5 \equiv 8 \pmod{13}; \\ (-4)a \equiv 8 \pmod{13} \end{array}\right\}$$

$$\Longleftrightarrow \{ 4a \equiv -8 \equiv 5 \pmod{13} \}$$

$$\Longleftrightarrow \{ (10)4a \equiv (10)(5) \pmod{13} \}$$

$$\Longleftrightarrow \{ 40a \equiv 50 \equiv 11 \pmod{13} \}$$

$$\Longleftrightarrow$$

Section 6: Solution to PROBLEM 2

Or equivalently,

$$\{ a \equiv 11 \pmod{13} \}.$$

We have shown that:

$$F(x) = 5x^{13} + 13x^{5} + 9ax \equiv 0 \pmod{13}$$
For all $x \in \mathbb{Z}$; $a \in \mathbb{Z}$

$$\Longleftrightarrow \quad \{ a \equiv 11 \pmod{13} \} \qquad (5)$$

It follows from (5), (4), and (3); that

$$\text{Set } A = \{ a \mid a \in \mathbb{Z},\ a \equiv 3 \pmod{5}, \text{ and } a \equiv 11 \pmod{13} \} \qquad (6)$$

By the definition of congruence,

$$\{ a \equiv 3 \pmod{5} \text{ and } a \equiv 11 \pmod{13} \}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} a = 3 + 5 \cdot k_1 = 11 + 13 \cdot k_2; \\ \text{for } k_1, k_2 \in \mathbb{Z} \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} a = 3 + 5k_1 = 11 + 13k_2; \\ \text{and } \quad 5k_1 = 13k_2 + 8; \\ \text{For } k_1, k_2 \in \mathbb{Z} \end{array} \right\} \qquad (7)$$

Section 6: Solution to PROBLEM 2

$$\left\{ \begin{array}{l} a = 3 + 5k_1 = 11 + 13k_2 \text{; and,} \\ 5k_1 = 13k_2 + 8; \\ \text{For } k_1, k_2 \in \mathbb{Z} \end{array} \right\} \quad (7)$$

From Result 5, it follows that since

$$5k_1 = 13k_2 + 8;$$

$$\left\{ \begin{array}{l} k_1 = 13t - 40 \text{ and } k_2 = 5t - 16; \\ \text{for some } t \in \mathbb{Z} \end{array} \right\} \quad (8)$$

Consequently, from (8) and (7); it follows that:

$$a = 3 + 5(13t - 40) = 65t + 3 - 200 = 65t - 197$$
$$= 13(5t - 16) + 11$$
$$= 65t - 208 + 11 = 65t - 197$$

We have proved that the set A described on page 14; consists precisely of the integers of the form $65t - 197$.

Section 6: Solution to PROBLEM2

$$A = \left\{ a \mid a \in \mathbb{Z},\ \text{and } a = 65t - 197;\ t \in \mathbb{Z} \atop t = 0, \pm 1, \pm 2, \dots \right\}$$

What is the smallest positive integer in A?

We set $a > 0 \Leftrightarrow 65t - 197 > 0$,

$$t > \frac{197}{65} = 3 + \frac{2}{65} > 3; \text{ hence,}$$

For $t = 4, 5, \dots$; $a > 0$

Thus, for $t = 4$ we have:

$$a = (65)(4) - 197 = 260 - 197 = \underline{\underline{63}}$$

The number $\boxed{63}$ is the smallest positive integer in the set A; and it is the answer to PROBLEM 2 ∎